Time-varying Risk and leverage effects: GARCH Models

Properties of financial time series
GARCH(p,q) models
Risk premium and ARCH-M models
Leverage effects and asymmetric GARCH models
ARCH/GARCH models are popular in empirical studies of finance

ARCH: Autoregressive Conditional Heteroscedasticity
Conditional variance is auto-correlated with lagged cond. Var. and lagged squared errors
ARCH invented by Engle (1982)
GARCH by Engle's student, Bollerslev (1986): generalized ARCH models, i.e. ARCH is a special case of GARCH models

Hereafter, we generally refer to "GARCH" models
But in the literature, Autoregressive Conditional Heteroscedasticity is still called "ARCH Effects", in honor of Engle.
8.1 Properties of financial time series

(1) excess kurtosis

Returns of financial asset exhibit that kurtosis $\geq 3$.
(\text{the kurtosis of normal dist. } = 3)
It means, the distribution of asset's returns is leptokurtic.
☑ Also referred to as thick tails, heavy tails, fat tails.

(2) volatility clustering

Returns in time series plots often show that "large changes tend to be followed by large changes, and small changes tend to be followed by small changes."
Fig. 8.1.1 Excess Kurtosis

leptokurtic

Fat tails
Fig. 8.1.2 Volatility Custering (IBM stock daily returns, 2004-09)

Large changes followed by large ones

Small changes followed by small ones
A GARCH(p,q) model has three components:

\[ r_t = f(x_t) + u_t \]  \hspace{1cm} (8.2.1)

\[ u_t = e_t \sqrt{h_t} \]  \hspace{1cm} (8.2.2)

\[ h_t = \omega + \sum_{i=1}^{q} \alpha_i u_{t-i}^2 + \sum_{i=1}^{p} \beta_i h_{t-i} \]  \hspace{1cm} (8.2.3)

- \( x_t \): independent variable(s)
- \( u_t \): residuals of "mean" equation
- \( h_t \): conditional variance

✓ Eq. (8.2.1) is called "mean equation."
✓ Eq. (8.2.3) is called "variance equation."
Normalized Residuals

Eq. (8.2.2) can be arranged to be:

\[ e_t = \frac{u_t}{\sqrt{h_t}} \quad (8.2.4) \]

\( e_t \) is normalized (standardized) residual

- It is often (but not necessarily) assume that \( e_t \sim N(0,1) \), i.e., \( e_t \) follows a normal distribution.
(Generalized) variance equation

\[ h_t = \omega + \sum_{i=1}^{q} \alpha_i u_{t-i}^2 + \sum_{i=1}^{p} \beta_i h_{t-i} \quad (8.2.3) \]

\( p, q \) are lagged orders of GARCH models.
That is, conditional variance, \( h_t \), is correlated to lagged squared residuals, \( u_{t-q}^2 \), and lagged conditional variances, \( h_{t-p} \).

- \( u_{t-q}^2 \) is called \textit{ARCH term}.
- \( h_{t-p} \) is called \textit{GARCH term}.
GARCH models in Empirical Studies

- **GARCH(1,1) models**

  \[ h_t = \omega + \alpha_1 u_{t-1}^2 + \beta_1 h_{t-1} \]  

  Eq. (8.2.2) connects the mean and variance equations.

  \[ u_t = e_t \sqrt{h_t} \]  

- **ARCH(q) models**

  Eq (8.2.3) with lagged order, \( p = 0 \), i.e., GARCH(0,q) models become ARCH(q) models.
8.2.1 Implications of GARCH models

Re-arranging eq (8.2.1) to be,

\[ u_t = r_t - f(x_t). \tag{8.2.6} \]

- **(Conditional) residuals mean that:**
  
  \( f(x_t) \) can explain part of variations in returns, \( r_t \), so that \( f(x_t) \) is expected. Thus,
  
  \( u_t \) is unexplained part of changes in returns

  - Also called "shock", news, or innovations
  - \( u_{t-1} \): previous (short-run) shock
  - Good news and bad news
    - \( u_{t-1} > 0 \) implies \( r_t > f(x_t) \). That is, actual return is higher than expected, \( f(x_t) \). This is essentially good news.
    
    Otherwise, \( u_{t-1} < 0 \) implies \( r_t < f(x_t) \), this is a bad news.
GARCH models' key is on the variance equation.

\[ h_t = \Omega + \alpha_1 u_{t-1}^2 + \beta_1 h_{t-1} \]

- **(1) new shock**

  \( h_t \) is a function of lagged squared residuals \( u_{t-1}^2 \).
  \( \alpha_1 \) is coefficient of new shocks on volatilities.

- **(2) persistence of volatility**

  The larger \( (\alpha_1 + \beta_1) \) is, the longer is the time that volatility persists.

- **(3) long-run (unconditional) variance**
Long-run (unconditional) variance

Weak stationarity assume that:
\[ E(h_t) = E(h_{t-1}) = \ldots = E(u_{t-1}^2) = \ldots = E(u_{t-q}^2) = \sigma^2 \]

Taking expectations of both sides of eq. (8.2.5) to have
\[ E(h_t) = \omega + \alpha_1 E(u_{t-1}^2) + \beta_1 E(h_{t-1}) . \]  \hspace{1cm} (8.2.7)

Substituting \( E(h_t) = E(h_{t-1}) = E(u_{t-1}^2) = \sigma^2 \) into (8.2.7) to obtain
\[ \sigma^2 = \omega + \alpha_1 \sigma^2 + \beta_1 \sigma^2 . \]  \hspace{1cm} (8.2.8)

Therefore, long-run (unconditional) variance is
\[ \sigma^2 = \frac{\omega}{1 - (\alpha_1 + \beta_1)} . \]  \hspace{1cm} (8.2.9)

Large \( \omega \) and small \((\alpha_1 + \beta_1)\) lead to large long-run (unconditional) variance.
### 8.3 ARCH Effects and models estimation

#### Table 8.3.1 Daily Stock Returns of Six US firms (%), 2004-2009.

<table>
<thead>
<tr>
<th></th>
<th>r_apple</th>
<th>r_bac</th>
<th>r_colac</th>
<th>r_disney</th>
<th>r_fedex</th>
<th>r_ibm</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Mean</strong></td>
<td>0.1518</td>
<td>-0.1098</td>
<td>-0.0016</td>
<td>0.0205</td>
<td>0.0137</td>
<td>0.0237</td>
</tr>
<tr>
<td><strong>Median</strong></td>
<td>0.1797</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>-0.0145</td>
<td>0.0126</td>
</tr>
<tr>
<td><strong>Max</strong></td>
<td>13.0190</td>
<td>30.2100</td>
<td>18.0650</td>
<td>14.8180</td>
<td>8.7678</td>
<td>10.8990</td>
</tr>
<tr>
<td><strong>Std. Dev.</strong></td>
<td>3.1853</td>
<td>4.5121</td>
<td>1.9987</td>
<td>1.9441</td>
<td>2.0689</td>
<td>1.4618</td>
</tr>
<tr>
<td><strong>C.V.</strong></td>
<td>20.9780</td>
<td>41.0800</td>
<td>1222.2000</td>
<td>94.9060</td>
<td>151.3900</td>
<td>61.7350</td>
</tr>
<tr>
<td><strong>Skewness</strong></td>
<td>-6.6501</td>
<td>-2.5387</td>
<td>-1.3891</td>
<td>0.6422</td>
<td>-0.2541</td>
<td>0.0689</td>
</tr>
<tr>
<td><strong>kurtosis</strong></td>
<td>143.41</td>
<td>49.05</td>
<td>25.60</td>
<td>8.90</td>
<td>5.18</td>
<td>6.07</td>
</tr>
</tbody>
</table>

*Note: Six firms include Apple, Bac, cola, Disney, Fedex, IBM.*
Fig. 8.3.1 Time series plot of daily returns, 2004-2009.
Table 8.3.2  Apple's daily returns in years 2004-2009.

<table>
<thead>
<tr>
<th>Year</th>
<th>2004</th>
<th>2005</th>
<th>2006</th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.45687</td>
<td>0.013694</td>
<td>0.067384</td>
<td>0.12511</td>
<td>-0.13848</td>
<td>0.44947</td>
</tr>
<tr>
<td>Median</td>
<td>0.2923</td>
<td>0.2827</td>
<td>-0.15758</td>
<td>0.21093</td>
<td>0.031558</td>
<td>0.25435</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>2.5629</td>
<td>4.897</td>
<td>2.3825</td>
<td>2.5799</td>
<td>3.6156</td>
<td>1.8618</td>
</tr>
<tr>
<td>C.V.</td>
<td>5.6097</td>
<td>357.6</td>
<td>35.357</td>
<td>20.62</td>
<td>26.11</td>
<td>4.1423</td>
</tr>
<tr>
<td>Skewness</td>
<td>1.0203</td>
<td>-10.568</td>
<td>0.74735</td>
<td>-0.53767</td>
<td>-0.38051</td>
<td>0.36256</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>3.3688</td>
<td>145.73</td>
<td>2.4594</td>
<td>2.2141</td>
<td>3.3957</td>
<td>0.69629</td>
</tr>
</tbody>
</table>

Next, how to test if there are "ARCH effects?"
Case 8.3.1  ARCH-LM tests

☑ Use AR(1) as the "mean" equation for r_twfi in FE-ex1.gdt
  r_twfi is Taiwan's monthly returns of financial stock index.
  The data file can be downloaded in http://yaya.it.cycu.edu.tw/gretl

Note: set sub-sample to 2000:01-2006:12

☐ 1. OLS estimation of AR(1) for r_twfi

☑ Generate returns variable: [add] □ [define new variable], the type "r_twfi=100*diff(ln(twfi))"

☑ In main menu, click [models] □ [Ordinary Least Squares] ,
☑ choose "r_twfi" as [dependent variable]
☑ click [lags] button , check [lags of dependent variable] , set it to 1 ;
  remove [const] in [dependent variables] , then click [OK] as in Fig.
  8.3.2
Case 8.3.1 ARCH-LM tests (cont.)

**Fig. 8.3.2 OLS estimation of AR(1)**

![Model output image]

Model 1: OLS, 使用中之子樣本範圍: 2000:01-2006:12 (T = 84) 
應變數 (Dependent variable): r_twfi  

<table>
<thead>
<tr>
<th>coefficient</th>
<th>std. error</th>
<th>t-ratio</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>r_twfi_1</td>
<td>-0.252767</td>
<td>0.106188</td>
<td>-2.380</td>
</tr>
</tbody>
</table>

Mean dependent var 0.001155  S.D. dependent var 0.091096  
Sum squared resid 0.644857  S.E. of regression 0.088144  
R-squared 0.063905  Adjusted R-squared 0.063905  
F(1, 83) 5.866191  P-value(F) 0.019586  
Log-likelihood 85.32998  Akaike criterion -168.6600  
Schwarz criterion -166.2291  Hannan-Quinn -167.6828  
rhô 0.029385  Durbin’s h 1.056548

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Case 8.3.1 ARCH-LM tests (cont.)

2. ARCH-LM test

- In menu of the estimated OLS model, click [Tests] → [ARCH],
- In [Lag order …], fill "1" at this time, for example.

Discussion

p-value = 0.6611, do not reject H0: No ARCH effects up to lag 1.
8.3.1 Estimation of GARCH models

☐ 下載「GJR-garchm」自訂函數套件

☑ Open "ex-IBM-2004-2009.gdt"
☑ Make sure to have access to Internet
Download the function package " GJR-garchm" from gretl server
That is, in menu of gretl, click [File] → [Function File] → [On server] as shown in Fig. 8.3.4。
Fig. 8.3.4 Download "GJR-garchm" function package.
Steps to estimate GARCH models

☐ (1) estimation of a model of mean eq.
☐ (2) testing if there are ARCH effects
☐ (3) Try GARCH(1,1) model first.
☐ (4) Testing ARCH effects left in standardized residual of models
☐ (5) Delete insignificant variables in mean eq.,
    then back to step 3, 4
☐ (6) Diagnosis of estimated models
Case 8.3.2    Estimating AR-GARCH models

This Case uses r_ibm in the data file "ex-IBM-2004-2009.gdt
Assume downloaded "GJR-garchm" function package

☐ mean eq.

Please use AR(3,25) models
(The key of case is on estimation of variance eq. of GARCH models)

\[
\begin{align*}
    r_{ibm} &= 0.0771 \times r_{ibm(-3)} + 0.0992 \times r_{ibm(-25)} + u \\
    [0.0025] & \quad [0.0001]
\end{align*}
\]

☑ Use OLS or ARIMA to estimate the models, then use Q test to examine residuals
☑ The result shown above is generated from [Exact maximum likelihood] under ARIMA in gretl. Q tests on u are shown in Fig. 8.3.5
Q tests on the residuals of AR(3,25) model suggest that Do not reject H0 「 no autocorrelation up to lag … 」.
1. Add lagged r_ibm to have r_ibm(-3) \( \cdot \) r_ibm(-25)

In gretl,
☑ click [r_ibm] variable,
☑ click [Add] \( \cdot \) [Lags of selected variable],
☑ In dialog window, fill "25" after [Number of lags to create]

As shown in Fig. 8.3.6.

2. Add "list"

In gretl click [Add] \( \cdot \) [define new variable], in what follows, enter

\[
\text{list } x0 = r_{ibm\,3} \quad r_{ibm\,25}
\]
Fig. 8.3.7 新增集合變數 x0。

```
list x0 = r_ibm_3 r_ibm_25
```

Fig. 8.3.6 新增 r_ibm 之後落後期至 25 期
3. 執行「GJR-garchm」自訂函數套件

- **click [File] » [Function File] 「On local…」**
- **double clicks on 「GJR-garchm」**
- **Fill the parameters as shown**

![Screenshot of GJR-garchm interface]

- It means to estimate GARCH(1,1)
- The list "x0" (Check here to save standardized residuals and h)
- Number of lags to do Q and Q² tests on residuals
- Check here to save standardized residuals and h
- Fill any "list" variable name

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You may also generate $x_0$ on the fly in "GJR-garchm" by click on beside the "indep" input box to have the following dialog window.
4. Q and $Q^2$ tests on standardized residuals of AR(3,25)-GARCH(1,1) models

Testing Results suggest: no autocorrelations & no ARCH effects left
5. Delete insignificant variables in mean eq, re-estimate GARCH(1,1) models

since coef. of \( r_{ibm}(-3) \), \( r_{ibm}(-25) \) are insignificant.

Re-estimate

\[
r_{ibm} = u
\]
\[
h_t = \omega + \alpha_1 u_{t-1}^2 + \beta_1 h_{t-1}
\]

☑ Only residuals in mean eq.

In Step 3, fill "null" in [Indep. Var] in 「 GJR-garchm 」

☑ Do Q and Q^2 tests on standardized resid. up to lag 10
Case 8.3.2  AR-GARCH models (discussion)

Estimation results

The estimated variance eq.:

$$h_t = 0.0703 + 0.1293u_{t-1}^2 + 0.8365 h_{t-1}$$

\[0.000\] \[0.000\] \[0.000\]

short-run impact coefficient = 0.1293;
persistence of volatility = 0.1293+0.8365 = 0.9658,

it suggests that any impact on volatility will persist for a long time.
The unconditional variance of $r_{ibm}$

$$= \frac{0.0703}{0.1293+0.8365} \approx 0.07277$$

Finally, plot the conditional variance and standardized residual, $h$ and $stz_u$ (appearing in main window of gretl) as shown in Fig in next page:
Time series plots of $h$ and $stz_u$